Numerical methods for solving differential equations

A baseball traveling through the air is acted upon by only two forces, namely gravity and the aerodynamic force. If the ball has little spin, the aerodynamic force is nearly all drag and acts opposite to the direction of velocity. The equation of motion is

$$\vec{F}_{gravity} + \vec{F}_{drag} = m\vec{a}$$

This is a vector equation and each vector has a horizontal and a vertical component. Because the drag is a non-linear function of velocity, $\left|\vec{F}_{drag}\right| \propto v^2$, we will not be able to find an analytic solution and a numerical solution will be necessary. Hence, we have two second-order non-linear differential equations. The usual technique is to replace the two-second order differential equations with four first-order equations. The four variables are the horizontal and vertical components of the position, x and y, and the velocity, v_x and v_y, respectively. The four equations are

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = \frac{\left|\vec{F}_{drag}\right| \cos q}{m}$$

$$\frac{dv_y}{dt} = \frac{\left|\vec{F}_{drag}\right| \sin q - g}{m}$$

where m is the mass of the ball and *?* is the angle between the velocity vector and the horizontal. Note that *?* changes continuously as the ball flies through the air. Euler's method is the simplest of the finite difference methods for integrating this set of differential equations. Although it is not recommended for practical use, it serves to illustrate the underlying strategy.

Consider the equation for dv_x/dt which can be rewritten as

$$\frac{dv_x}{dt} = f\left(t, v_x\right).$$

If we know the horizontal component of the ball's velocity at the moment it leaves the bat, $t = 0$, then we can evaluate the tangent to the solution at this point by evaluating $f\left(t, v_x\right)$. We advance the solution by moving in the direction of the tangent by a small amount. This is the time step input used in the program. If the change in t is ? t, then the change in v_x along the tangent direction is given by $\Delta v_x = f\left(t, v_x\right)\Delta t$. This yields a new point on an approximate solution to the differential equation. This process is repeated until the desired value is reached. Trajectory.exe terminates when the vertical position drops below the initial altitude.

The algorithm can be schematically represented as:

Initial $(\boldsymbol{t}, \boldsymbol{v_x})$

$\boldsymbol{v_x} = \boldsymbol{v_x} + \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{v_x})\Delta \boldsymbol{t}$

$\boldsymbol{t} = \boldsymbol{t} + \Delta \boldsymbol{t}$

Repeat

Euler’s method is not very accurate nor is it stable. *Trajectory.exe* uses a more sophisticated algorithm called a fourth-order Runge-Kutta. Details of this algorithm may be found in *Numerical Recipes*.